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B. Sc. Sem-II

Leibnitz's theorem :- (The nth derivative of product of two functions)
Let u & v are two function of x possessing derivatives of the nth order then

$$(uv)_n = {}^nC_0 \cdot u_n v + {}^nC_1 \cdot u_{n-1} v_1 + {}^nC_2 u_{n-2} v_2 + \cdots + {}^nC_r u_{n-r} v_r + {}^nC_n u \cdot v_n .$$

Here suffixes denote their differentiation and ${}^nC_0, {}^nC_1 \cdots {}^nC_r \sim {}^nC_n$ are binomial co-efficient

i.e $\quad {}^nC_r = \underline{|n} / \underline{|r}\,\underline{|n-r}$

Que :- (1) Find nth derivative of $e^x(2x+3)^3$

Sol'n :- Let $u = e^x$, $v = (2x+3)^3$

So that $v_1 = 3(2x+3)^2 \cdot 2$

$v_2 = 12(2x+3) \cdot 2 = 24(2x+3)$

$v_3 = 48$

By Leibnitz's theorem

$$\left[e^x (2x+3)^3\right]_n = (e^x)_n (2x+3)^3 + {}^nC_1 (e^x)_{n-1} (2x+3)^3_1 + {}^nC_2 (e^x)_{n-2} (2x+3)^3_2 + {}^nC_3 (e^x)_{n-3} (2x+3)^3_3$$

$$= e^x\left[(2x+3)^3 + n \cdot 6(2x+3)^2 + \frac{n(n-1)}{\underline{|2}} 2x(2n+3) + \frac{n(n-1)(n-2)}{\underline{|3}} \times 48\right]$$

Proved.

Que 2. Find the nth derivative of (i) $x^2 \log 3x$ (ii) $x^2 \cos x$

(i) Let $u = \log 3x$, $v = x^2 \Rightarrow v_1 = 2x$, $v_2 = 2$

By Leibnitz's theorem

$$(uv)_n = {}^nC_0 u_n v + {}^nC_1 \cdot u_{n-1} v_1 + {}^nC_2 \cdot u_{n-2} v_2 + \cdots$$

$= (\log 3x)_n \cdot x^2 + n(\log 3x)_{n-1} \cdot 2x + \frac{n(n-1)}{\underline{|2}} \cdot (\log 3x)_{n-2}$

$= \frac{(-1)^{n-1} \cdot \underline{|n-1} \cdot 3^n}{(3x)^n} \cdot x^2 + n \cdot \frac{(-1)^{n-1} \cdot \underline{|n-2} \cdot 3^{n-1}}{(3x)^{n-1}} \cdot 2x$

$+ \frac{n(n-1)(-1)^{n-3} \cdot \underline{|n-3} \cdot (3)^{n-2}}{(3x)^{n-2}}$

$= \frac{(-1)^{n-3} \underline{|n-3}}{x^n} \left[(n-1)(n-2)x^2 - \frac{n(n-2)}{x} + n(n-1) \right]$

$= 0.$

(ii) $x^2 \cos x$, $u = \cos x$, $V = x^2$, $V_1 = 2x$, $V_2 = 2$

$(x^2 \cos x)_n = (\cos x)_n \cdot x^2 + n(\cos x)_{n-1} \cdot 2x + \frac{n(n-1)}{\underline{|2}} (\cos x)_{2-2} \cdot 2$

$= \cos\left(x + \frac{n\pi}{2}\right) \cdot x^2 + n \cdot \cos\left(x + \frac{(n-1)\pi}{2}\right) \cdot 2x$

$+ \frac{n(n-2)}{\underline{|2}} \cdot \cos\left(x + \frac{(n-2)\pi}{2}\right) \cdot 2$

$= \cos\left[x + \frac{n\pi}{2}\right] x^2 + n \cos\left(n + \frac{(n-1)\pi}{2}\right) \cdot 2x$

$+ n(n-1) \cdot \cos\left(n + \frac{(n-2)\pi}{2}\right]$

Que:— If $y = (2-3x)^{10}$. Find y_9

Sol^n:— $y = (-1)^{10} \cdot (3x-2)^{10}$

$y = (3x-2)^{10}$ $\quad m = 10, \; a = 3 \; b = -2$

therefore of $y = (ax+b)^m$

$y_n = m(m-1) \cdots (m-n+1) \cdot a^n \cdot (ax+b)^{m-n}$

put $n = 9$ $\quad m = 10 \quad a = 3, \; b = -2$

$\Rightarrow y_9 = 10 \cdot 9 \cdot 8 \cdots (10-9+1) \cdot (3)^9 \cdot (3x-2)$

$= \underline{|10} \cdot 3^9 (3x-2)$